Groups Ia Practice Sheet A

Michaelmas 2014 Julia Goedecke

These questions are not supposed to form the work for one of the regular 4 groups supervisions, but instead they give you opportunities to practise getting used to axioms and definitions in your own time. If you find this useful, try to make similar questions for yourself on later material of the course.

Examples of groups

1. Show that $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ (rational numbers without zero) is a group under multiplication. [This means that you should use standard multiplication as the operation for the group.] Is $\mathbb{R}^+$ (positive real numbers) a group under multiplication?

2. Show that $\mathbb{R}^3$ with (componentwise) addition (you might know it as vector addition) is a group. What is the inverse to a general vector (x, y, z)?

3. Do the natural numbers $\mathbb{N}$ (with or without zero, choose which you prefer) form a group under addition? If yes, show carefully that all axioms hold. If no, show which axioms do not hold.

4. Is $\mathbb{Q}$ a group under multiplication? Again prove your answer carefully.

5. Show that $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ does not form a group under division. Which axioms do not hold? [You should in particular pay attention to the identity axiom.]

6. On the integers $\mathbb{Z}$ define the operation $n * m = n + m + 1$. Show that $\mathbb{Z}$ forms a group under this operation. What is the identity element? Show that $-(n + 2)$ is the inverse to n. Is the group abelian?

7. Show that composition of functions is always an associative operation. What extra properties might you need to get a group with composition as the group operation?

8. * *This is a more involved question, but still very interesting to try.*

   Let X be a set. The **powerset** $\mathcal{P}(X)$ is the set of all subsets of X:

   $$\mathcal{P}(X) = \{A \mid A \subseteq X\}$$

   Does $\mathcal{P}(X)$ form a group under the operation $\cap$ (intersection)? Does $\mathcal{P}(X)$ form a group under the operation $\cup$ (union)?

   The **symmetric difference** of two subsets $A, B \subseteq X$ is the set of elements which are in *exactly one* of A and B:

   $$A \triangle B = (A \cup B) \setminus (A \cap B).$$

   Show that $\mathcal{P}(X)$ forms a group under $\triangle$. What is the identity element? For now, assume that symmetric difference is an associative operation. You will meet a nice way of proving it later in Numbers and Sets.

Groups Ia Practice Sheet B

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Properties of general groups

In the following questions, let G be a group with operation $*$.

1. Without looking at your lecture notes, show that inverses are unique. That is, if $g, h \in G$ with $h * g = e = g * h$, show that $h = g^{-1}$.

2. Show that the equation $a * x = b$ has a unique solution for x in G, and find this solution.

3. For $a, b, c, d \in G$, use the associativity axiom to show that $((a * b) * c) * d = a * (b * (c * d))$. You will find similarly that all possible ways to bracket this product of four elements gives the same answer. Does this extend to products of five elements?

4. What is the inverse of $a * b * a^{-1}$? Simplify the expression $(a * b * a^{-1})^n$ as much as you can. [Here "to the power n" just means multiply (or "star") the expression in the brackets n times with itself.]

Subgroups

5. Show that the even numbers $2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\}$ form a subgroup of $\mathbb{Z}$. Show also that for any $n \in \mathbb{Z}$, the set $n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}$ forms a subgroup of $\mathbb{Z}$.

6. (a) Show that the rotations of a regular triangle form a subgroup of all symmetries of the triangle. (You can also do this for a different or general n-gon if you wish.)
   (b) Show that the symmetries of a regular triangle form a subgroup of the symmetries of a regular hexagon. (How many of such subgroups are there in the symmetries of a regular hexagon?)

7. Show that $\{(x, 0, 0) \mid x \in \mathbb{R}\}$ is a subgroup of $\mathbb{R}^3$ (with addition). Show also that the sets $\{(x, x, x) \mid x \in \mathbb{R}\}$ and $\{(x, y, 0) \mid x, y \in \mathbb{R}\}$ are subgroups. Why is $\{(x, 1, 2) \mid x \in \mathbb{R}\}$ not a subgroup?

Groups Ia Practice Sheet C

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Homomorphisms

1. Show that $f\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ given by $f(n) = 2n$ is a group homomorphism. Similarly show $f_k\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ with $f_k(n) = kn$ is a group homomorphism. Can you think of any others?
2. Show that $f\colon \mathbb{R} \longrightarrow \mathbb{R}^3$ with $f(x) = (x, x, x)$ and $g\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ with $g((x, y, z)) = x$ are group homomorphisms. What about $h\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ with $h((x, y, z)) = x + y + z$? Can you find any more similar group homomorphisms?
3. Recall that $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$ forms a group under multiplication. Show that $f\colon \mathbb{Q}^* \longrightarrow \mathbb{Q}^*$ with $f(\frac{a}{b}) = \frac{b}{a}$ is a group homomorphism. A similar idea with $g\colon G \longrightarrow G$ defined by $g(a) = a^{-1}$ only works for abelian groups! (Why?)
4. Show that $f\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ with $f(n) = n + 1$ is *not* a group homomorphism.
5. Show that the only constant function $f\colon G \longrightarrow H$ between to groups which is a group homomorphism is the one defined by $f(a) = e$ for all $a \in G$.
6. Let $f\colon G \longrightarrow H$ be a group homomorphism. Show that $f(g^{-1}) = f(g)^{-1}$. [Remember that inverses are unique!]

Groups Example Sheet 1

Michaelmas 2014 Julia Goedecke

Please send comments and corrections to jg352.

1. Let G be any group. Show that the identity e is the unique solution of the equation $a^2 = a$.

2. Let H_1 and H_2 be two subgroups of the group G.

   Show that the intersection $H_1 \cap H_2$ is a subgroup of G.

   Show that the union $H_1 \cup H_2$ is a subgroup of G if and only if one of the H_i contains the other.

3. Show that the set of functions on $\mathbb{R}$ of the form $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$, forms a group under composition of functions. Is this group abelian?

4. Let G be a finite group.

   (a) Let $a \in G$. Show that there is a positive integer n such that $a^n = e$, the identity element. (The least such positive n is the *order* of a.)

   (b) Show that there exists a positive integer n such that $a^n = e$ for all $a \in G$. (The least such positive n is the *exponent* of G.)

5. Show that the set G of complex numbers of the form $\exp(i\pi t)$ with t rational is a group under multiplication (with identity 1). Show that G is infinite, but that every element a of G has finite order.

6. Let S be a finite non-empty set of non-zero complex numbers which is closed under multiplication. Show that S is a subset of the set $\{z \in \mathbb{C} : |z| = 1\}$. Show that S is a group, and deduce that for some $n \in \mathbb{N}$, S is the set of n-th roots of unity; that is, $S = \{\exp(2k\pi i/n) : k = 0, \ldots, n-1\}$.

7. Let $G = \{x \in \mathbb{R} : x \neq -1\}$, and let $x * y = x + y + xy$, where xy denotes the usual product of two real numbers. Show that $(G, *)$ is a group. What is the inverse 2^{-1} of 2 in this group? Solve the equation $2 * x * 5 = 6$.

8. Let G be a group in which every element other than the identity has order two. Show that G is abelian. Show also that if G is finite, the order of G is a power of 2. [Consider a minimal generating set.]

9. Let G be a group of even order. Show that G contains an element of order two.

10. Let G be a finite group and f a homomorphism from G to H. Let $a \in G$. Show that the order of $f(a)$ is finite and divides the order of a.

11. Show that the dihedral group D_{12} is isomorphic to the direct product $D_6 \times C_2$. Is D_{16} isomorphic to $D_8 \times C_2$?

12. How many homomorphisms $D_{2n} \longrightarrow C_n$ are there? How many isomorphisms $C_n \longrightarrow C_n$?

Groups Example Sheet 2

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Please send comments and corrections to jg352.

1. Write these permutations as products of disjoint cycles and compute their order and sign:
   (a) $(12)(1234)(12)$;
   (b) $(123)(235)(345)(45)$.

2. What is the largest possible order of an element in S_5? And in S_9?
   Show that every element in S_{10} of order 14 is odd.

3. Let G be a subgroup of the symmetric group S_n. Show that if G contains any odd permutations then precisely half of the elements of G are odd.

4. (a) Show that the symmetric group S_4 has a subgroup of order d for each divisor d of 24, and find two non-isomorphic subgroups of order 4.
   (b) Show that the alternating group A_4 has a subgroup of each order up to 4, but there is no subgroup of order 6.

5. A finite group G is generated by a set T of elements of G if each element of G can be written as a finite product (possibly with repetitions) of powers of elements of T. Show that the symmetric group S_n is generated by each of the following sets of permutations:
   (a) the set $\{(j, k) : 1 \leq j < k \leq n\}$ of all transpositions in S_n;
   (b) the set $\{(j, j + 1) : 1 \leq j < n\}$;
   (c) the set $\{(1, k) : 1 < k \leq n\}$;
   (d) the set $\{(1, 2), (12 \ldots n)\}$ consisting of a transposition and an n-cycle.

6. Let H be a subgroup of the group G. Find a (natural) bijection between the set of all left cosets and the set of all right cosets of H in G.

7. Show that if a group G contains an element of order six, and an element of order ten, then G has order at least 30.

8. Let H be a subgroup of the (finite) group G, let K be a subgroup of H. Show that the index $|G : K|$ equals the product $|G : H||H : K|$.

9. Show that the set $\{1, 3, 5, 7\}$ with multiplication modulo 8 is a group. Is this group isomorphic to C_4 or $C_2 \times C_2$? Justify your answer.

10. Let G be a group. If H is a normal subgroup of G and K is a normal subgroup of H, is K a normal subgroup of G?

11. Let K be a normal subgroup of index m in the group G. Show that $a^m \in K$ for any $a \in G$.

12. Show that any group of order 10 is either cyclic or dihedral.

13. Let $D_{12} = \langle r, s \mid r^6 = e = s^2, rs = sr^{-1}\rangle$ be the dihedral group of order 12.
    (a) Find all subgroups of D_{12}. Which of them are normal? [There are 16 subgroups in total.]
    (b) For each proper normal subgroup N of D_{12}, determine what standard group the quotient D_{12}/N is isomorphic to.

14. Consider a pack of $2n$ cards, numbered from 0 to $2n-1$. An *outer perfect shuffle* is a shuffle of the cards, in which one first splits the pack in two halves of equal sizes and then interleaves the cards of the two halves in such a way that the top and bottom card remain in the top and bottom position. Show that the order of the outer shuffle is the multiplicative order of 2 modulo $2n-1$.

    Deduce that after at most $2n-2$ repetitions of the outer shuffle we get the cards in the pack into the original position.

    What is the actual order of the outer shuffle of the usual pack of 52 cards?

    (There is also an *inner perfect shuffle* which differs from the outer shuffle in that the interleaving of the cards of the two halves is done so that neither the top nor the bottom card remains in the same position. What is the order of this shuffle of the usual pack of 52 cards?)

Groups Example Sheet 3

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Please send comments and corrections to jg352.

1. Let D_{2n} be the group of symmetries of a regular n-gon. Show that **any** subgroup K of rotations is normal in D_{2n}, and identify the quotient D_{2n}/K. (Identify means: what standard group is it isomorphic to?)

2. Show that D_{2n} has two conjugacy classes of reflections if n is even, but only one if n is odd.

3. Let Q be a plane quadrilateral. Show that its group $G(Q)$ of symmetries has order at most 8. For each n in the set $\{1, 2, \ldots, 8\}$, either give an example of a quadrilateral Q with $G(Q)$ of order n, or show that no such quadrilateral can exist.

4. List all the subgroups of the dihedral group D_8, and indicate which pairs of subgroups are isomorphic.
   Repeat for the quaternion group Q_8.

5. Find the conjugacy classes of D_8 and their sizes. Show that the centre Z of the group has order 2, and identify the quotient group D_8/Z of order 4.
   Repeat with the quaternion group Q_8.

6. What is the group of all rotational symmetries of a Toblerone box, a solid triangular prism with an equilateral triangle as a cross-section, with ends orthogonal to the longitudinal axis of the prism? And the group of all symmetries?

7. Suppose that the group G acts on the set X. Let $x \in X$, let $y = g(x)$ for some $g \in G$. Show that the stabiliser G_y equals the conjugate gG_xg^{-1} of the stabiliser G_x.

8. Let G be a finite group and let X be the set of all subgroups of G. Show that G acts on X by $g : H \longmapsto gHg^{-1}$ for $g \in G$ and $H \in X$, where $gHg^{-1} = \{ghg^{-1} : h \in H\}$. Show that the orbit containing H in this action of G has size at most $|G|/|H|$. If H is a proper subgroup of G, show that there exists an element of G which is contained in no conjugate gHg^{-1} of H in G.

9. Let G be a finite group of prime power order p^a, with $a > 0$. By considering the conjugation action of G, show that the centre Z of G is non-trivial.
   Show that any group of order p^2 is abelian, and that there are up to isomorphism just two groups of that order for each prime p.

10. Find the conjugacy classes of elements in the alternating group A_5, and determine their sizes. Show that A_5 has no non-trivial normal subgroups (so A_5 is a *simple* group).
    Show that if H is a proper subgroup of index n in A_5 then $n > 4$. [Consider the left coset action of A_5 on the set of left cosets of H in A_5.]

Groups Example Sheet 4

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1. Let G be the set of all 3×3 matrices of the form
$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$
with $x, y, z \in \mathbb{R}$. Show that G is a subgroup of the group of invertible real matrices under multiplication. Let H be the subset of G given by those matrices with $x = z = 0$. Show that H is a normal subgroup of G and identify G/H. (This G is called the *Heisenberg group*.)

2. Take the Heisenberg group as above, but this time with entries in $\mathbb{Z}_3$. Show that every non-identity element of this group has order 3, but the group is not isomorphic to $C_3 \times C_3 \times C_3$.

3. Recall that the *centre* of a group G consists of all those elements of G that commute with all the elements of G. Show that the centre Z of the general linear group $GL_2(\mathbb{C})$ consists of all non-zero scalar matrices. Identify the centre of the special linear group $SL_2(\mathbb{C})$.

4. Consider the set of matrices of the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & t \end{smallmatrix}\right)$ for $t \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Show that these form a group under matrix multiplication. More generally, show that if a set of matrices forms a group under multiplication, then either all matrices in the set have non-zero determinant, or all have zero determinant.

5. Let G be the set of all 3×3 real matrices of determinant 1 of the form
$$\begin{pmatrix} a & 0 & 0 \\ b & x & y \\ c & z & w \end{pmatrix}.$$
Verify that G is a group. Find a homomorphism from G onto the group $GL_2(\mathbb{R})$ of all non-singular 2×2 real matrices, and find its kernel.

6. Let K be a normal subgroup of order 2 in the group G. Show that K lies in the centre of G; that is, show $kg = gk$ for all $k \in K$ and $g \in G$.

   Exhibit a surjective homomorphism of the orthogonal group $O(3)$ onto C_2 and another onto the special orthogonal group $SO(3)$.

7. Consider the Möbius maps $f(z) = e^{2\pi i/n} z$ and $g(z) = 1/z$. Show that the subgroup $G = \langle f, g \rangle$ of the Möbius group $\mathcal{M}$ is a dihedral group of order $2n$.

8. Let $g(z) = (z + 1)/(z - 1)$. By considering the points $g(0)$, $g(\infty)$, $g(1)$ and $g(i)$, find the image of the real axis $\mathbb{R}$ and of the imaginary axis $\mathbb{I}$ under g. What is $g(\Sigma)$, where Σ is the first quadrant in $\mathbb{C}$?

9. What is the order of the Möbius map $f(z) = iz$? If h is any Möbius map, find the order of hfh^{-1} and its fixed points. Use this to construct a Möbius map of order four that fixes 1 and -1.

10. Let G be the group of Möbius transformations which map the set $\{0, 1, \infty\}$ onto itself. Find all the elements in G. To which standard group is G isomorphic? Justify your answer.

    Find the group of Möbius transformations which map the set $\{0, 2, \infty\}$ onto itself. [Try to do as little calculation as possible.]

11. Let G be as in the previous question. Show that, given $\sigma \in S_4$, there exists $f_\sigma \in G$ for which, whenever z_1, z_2, z_3 and z_4 are four distinct points in $\mathbb{C}_\infty$, we have $f_\sigma([z_1, z_2, z_3, z_4]) = [z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}]$. [You may want to start with σ a transposition in S_4.]

    Show that the map $\sigma \longmapsto f_{\sigma^{-1}}$ from S_4 to G gives a homomorphism from S_4 onto S_3. Find its kernel.

12. Let G be the special linear group $SL_2(5)$ of 2×2 matrices of determinant 1 over the field $\mathbb{F}_5$ of integers modulo 5, so that the arithmetic in G is modulo 5. Show that G is a group of order 120. Prove that $-I$ is the only element of G of order 2.

    * Find a subgroup of G isomorphic to Q_8, and an element of order 3 normalising it in G. Deduce that G has a subgroup of index 5, and obtain a homomorphism from G to S_5. Deduce that $SL_2(5)/\{\pm I\}$ is isomorphic to the alternating group A_5.

    Show that $SL_2(5)$ has no subgroup isomorphic to A_5.